

Black hole solutions of gravity theories with nonminimal coupling between matter and curvature

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We study black hole solutions in an extension of General Relativity (GR) with an explicit non-minimal coupling between matter and curvature. General black hole solutions satisfying the known energy conditions are derived including the ones with anti-de Sitter background. These solutions differ from those of GR just by a coupling function dependent rescaling of the mass and charge of the black hole and by a “dressing” of the cosmological constant. The existence of black hole solutions of the nonminimally coupled theory as well as the conditions for a suitable weak field limit are considered as a constraint on the coupling function responsible for the nonminimal coupling between matter and curvature. The “dressing” of the cosmological constant is then used to address the cosmological constant problem.

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I. INTRODUCTION

As far as it is known General Relativity (GR) describes accurately all gravitational phenomena at the solar system level [1, 2] and it gives rise to striking predictions among which black hole solutions are possibly the most bizarre and surprising ones. However, at galactic and cosmological scales, in order to account for the observations two unknown components must be introduced: dark matter, to explain the rotation curve of spiral galaxies and the dynamical mass of clusters; and dark energy, an exotic form smooth distribution of energy to account the late-time accelerated expansion of the Universe. Strikingly, these two dark components constitute about 95% of the energy content of the Universe and their nature remains still a mystery.

Of course, several alternative gravitational theories have been proposed to account for the observations usually explained by the presence of dark matter and dark energy, such as, for instance, $F(R)$ theories of gravity [3–5]. Another interesting alternative admits a non-minimal coupling between curvature and matter [6] (see Refs. [7, 8] for proposals in the context of cosmology). The later is characterized by a coupling function $f(R)$ between matter and the curvature and has a wide range of theoretical and observational implications. It has bearings on issues such as stellar stability [9, 10], preheating after inflation [11], mimicking of dark matter in galaxies [12] and clusters [13] and the large scale effect of dark energy [14]. The issue of energy density perturbations in the context of an homogeneous and isotropic cosmological background was studied in Ref. [15]. For a complete and updated review of the main developments involving this type of gravity model with nonminimal coupling between matter and curvature, the reader is referred to Ref. [16].

In this work, we examine static, spherically symmetric black hole solutions in the context of theories with non-minimal coupling between matter and curvature. The existence and the properties of these solutions is an important test for all alternative theories of gravity. Of particular relevance is the deviation of the black solution from the usual ones of GR, which is expected to provide useful information about the strong coupling regime of the theory.

As we shall see, the study of black hole solutions of non-minimally coupled (NMC) gravity theories has a rich set of implications once the presence of various matter sources is considered (so to test the coupling term) and once non-trivial backgrounds are admitted. As expected, a relevant matter when studying these solutions is whether they saturate or not the null-energy condition (NEC). The solutions that saturate the NEC, which will be discussed at length in this paper, represent the generalization to the non-minimally coupled theory of the well-known Schwarzschild and Reissner-Nordstrom solutions (both in flat and anti de Sitter space) of GR.

Owing to the non-minimal coupling between matter sources and curvature, one naively expects strong deviation of these solutions from their behaviour in GR. Most particularly in what concerns their uniqueness. As we shall see, our investigation does not confirm this naive expectation. The generalization of the Schwarzschild and Reissner-Nordstrom solutions are very similar to their GR counterparts (they differ just by a coupling function dependent rescaling of their mass and charge and by a dressing of the cosmological constant). Moreover, we will find strong evidence that they are also essentially unique.

Another important point when dealing with gravity theories with non-minimal coupling between matter and curvature is the issue of the determination of the coupling function $f(R)$. The NMC theory is regarded as an effective theory with the coupling function $f(R)$ parametrizing our ignorance about a presumed fundamental theory of gravity. On the other hand, the local behaviour of the coupling function $f(R)$ encodes the physical information on the behaviour of the NMC theory at length scales $\sim R^{-1/2}$. Within this phenomenological perspective it is very important to gather information coming from different length scales and coupling regimes of gravity, so to constrain the analytic form of $f(R)$.

Until now most of the constraints on the coupling function came from the large scale, infrared (IR), behaviour of gravity [12–14, 16]. On its hand, the investigation on black holes is expected to provide insight not only about the IR behaviour of $f(R)$ (through the weak field limit) but, owing to the presence of singularities, also on its ultraviolet (UV) behaviour. We shall show that this is the case. Investigation of the black hole solutions will allow us to establish constraints on the local behaviour of $f(R)$ at the black hole singularity and at the flat asymptotic region. As a byproduct of our research we consider the dressing of the cosmological constant to examine the possibility to generate a natural hierarchy of mass scales between the bare and the dressed cosmological constant.

This work is organised as follows. In the next section we review the main features of the non-minimally coupled matter-curvature model [6] that we have in mind. In section III, we examine the energy conditions suitable to study black hole type solutions, most particularly NEC. We analyze then static, spherically symmetric solutions of the alternative theory of gravity under the conditions that the trace of the energy momentum tensor is constant and that the components satisfy the condition $T_0^0 = T_1^1$ (sections IV and V). How natural are these conditions will be then discussed. Next, de Sitter and anti-de Sitter backgrounds are considered once a cosmological constant is introduced (section VI). Solutions in these backgrounds are studied in section VII. In section VIII, charged black hole like solutions are sought in various backgrounds and under the conditions set by the NEC. In the remaining sections we study the weak field limit (section IX), the constraints that the existence of the considered solutions and the weak field limit pose on the coupling function (section X) and use the freedom in choosing the coupling function to address the cosmological constant problem (section XI). Finally, section XII contains our conclusions.

II. FIELD EQUATIONS AND STRESS ENERGY TENSOR

In this paper we consider a gravity theory in which the coupling between the gravitational field and matter is non-minimal. Invariance of the action under diffeomorphisms together with the requirement that in the flat limit the Lagrangian for the matter fields \mathcal{L}_m reduces to that in Minkowski space, fixes the dependence of \mathcal{L}_m on the metric tensor $g_{\mu\nu}$. On the other hand, these requirements leave open the possibility that \mathcal{L}_m enters in the action not in the minimal way (multiplied by the covariant volume element) but nonminimally, through an arbitrary function of the scalar curvature.

We are therefore lead to consider for the matter-gravity coupled system an action of the form [6]:

$$S = \int \left\{ \frac{1}{2}R + [1 + \lambda f(R)]\mathcal{L}_m \right\} \sqrt{-g}d^4x, \quad (2.1)$$

where $f(R)$ is an arbitrary function of the curvature scalar, we have chosen units such that $8\pi G = 1$ and λ is a constant parameter. As mentioned in the introduction, the gravity theory described by the action (2.1) has been investigated given its interesting cosmological implications and the possibility for a gravitational solution for the dark matter [12, 13] problem and the large scale effect of dark energy [14].

Varying the action with respect to the metric $g^{\mu\nu}$, we get the field equations for the gravity field,

$$G_{\mu\nu} = -2\lambda f_R \mathcal{L}_m R_{\mu\nu} + 2\lambda(\nabla_\mu \nabla_\nu - g_{\mu\nu} \square)\mathcal{L}_m f_R + [1 + \lambda f(R)]T_{\mu\nu}, \quad (2.2)$$

where $G_{\mu\nu}$ is the Einstein's tensor, $f_R(R) = \frac{df}{dR}$ and $T_{\mu\nu}$ is the energy-momentum tensor,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathcal{L}_m}{\delta g^{\mu\nu}}. \quad (2.3)$$

By contracting Eq. (2.2) with $g^{\mu\nu}$ we get the general form for the scalar curvature:

$$R = \frac{-6\Box(\lambda \mathcal{L}_m f_R) + [1 + \lambda f(R)]T}{2\lambda \mathcal{L}_m f_R - 1}. \quad (2.4)$$

Thus, due to the nonminimal coupling between scalar curvature and matter, R is no longer proportional to the trace of the energy-momentum tensor, but it also depends on the higher order derivative of the coupling term.

Also the field equations for the matter fields, obtained by varying the action (2.1) with respect to the matter fields will get a dependence from the scalar curvature and its derivatives. In this paper we are mainly concerned in the dynamics of gravity. In the following we will, therefore, use the field equations for matter fields only when needed to solve the dynamics of the gravitational field.

In the following we will need an expression relating the higher order derivative coupling term and the first two components of the energy-momentum tensor. Some algebraic manipulation renders from the field equations (2.2)

$$\frac{[1 + 2\lambda f_R \mathcal{L}_m]}{[1 + \lambda f]} (R_0^0 - R_1^1) + \frac{[\nabla_1 \nabla^1 - \nabla_0 \nabla^0](2\lambda \mathcal{L}_m f_R)}{[1 + \lambda f]} = T_0^0 - T_1^1. \quad (2.5)$$

The nonminimal nature of the gravity-matter coupling is fully evident in the RHS of Eq. (2.2). Notice that the curvature appears also as source of the gravitational field so that $T_{\mu\nu}$ is not covariantly conserved as in GR but satisfies the equation

$$\nabla^\mu T_{\mu\nu} = \frac{\lambda f_R}{1 + \lambda f} [g_{\mu\nu} \mathcal{L}_m - T_{\mu\nu}] \nabla^\mu R, \quad (2.6)$$

as one can easily verify taking into account the contracted Bianchi identities, $\nabla^\mu G_{\mu\nu} = 0$ and the following identity:

$$(\Box \nabla_\nu - \nabla_\nu \Box) f_R = R_{\mu\nu} \nabla^\mu f_R. \quad (2.7)$$

Physically, equation (2.6) describes an exchange of energy and momentum between matter and higher order derivative curvature terms. An important point to consider are the conditions to be satisfied in order that this exchange of energy and momentum does not take place. One can easily see that the covariant derivative of $T_{\mu\nu}$, turns to zero only in two cases (apart from the obvious case $f(R) = \text{const.}$, corresponding to GR).

1. $T_{\mu\nu} = g_{\mu\nu} \mathcal{L}_m$

2. $R = \text{const.}$

In the first case, \mathcal{L}_m does not depend explicitly on the metric. This corresponds to terms in the matter Lagrangian in which the coupling to gravity is only through the covariant volume (e.g a term $\sqrt{-g} V(\phi)$ for a scalar field or a cosmological constant). The most interesting physical case belonging to this class is represented by a cosmological constant,

$$\mathcal{L}_m = -\Lambda, \quad (2.8)$$

which in GR gives origin to maximally symmetric spaces: Minkowski, de Sitter (dS) and anti-de Sitter (AdS), corresponding respectively to vanishing, positive and negative Λ .

The second case also admits as solutions the maximally symmetric spaces. The other interesting cases belonging to this class are those spaces with constant scalar curvature, which are not maximally symmetric. In GR they describe vacuum solutions ($T_{\mu\nu} = 0$) like the Schwarzschild (SCHW) solution or, conformal invariant matter ($T = T^\mu_\mu = 0$) like the Reissner-Nordstrom (RN) solution.

The extension of the above-mentioned solutions of GR to the case of nonminimally coupled gravity theories is the main goal of this paper. Here, we just point out that this extension is far from being trivial and in general it requires non trivial constraints on the form of the coupling function $f(R)$. This is quite evident from the form of the field equations (2.2) and from the expression of the scalar curvature (2.4). Differently from GR, in the RHS of these equations enters the scalar curvature, through the coupling function $f(R)$. For instance, we see from equation (2.4) that a traceless matter stress-energy tensor does not automatically imply vanishing of the scalar curvature of the space-time. Additional constraints on the form of the coupling function $f(R)$ (or on the form of \mathcal{L}_m) are needed in order to achieve that.

For the same reason, the question whether the existence of other solutions, apart from the GR solutions, is clearly nontrivial. One could consider for instance solutions with scalar hair. However, in this paper we will limit our investigation to the extension of the well-known GR solutions (maximally symmetric spaces, the Schwarzschild and Reissner-Nordstrom solutions, both in Minkowski and AdS space).

III. ENERGY CONDITIONS

Energy conditions in GR, and in particular the the null-energy condition (NEC) and the strong-energy condition (SEC), correspond to a fairly general way to translate geometric informations on the congruence of geodesics into conditions for the matter stress energy tensor. In the context of a nonminimally coupled gravity theory, one naturally expects the energy conditions to be drastically modified by the presence of the curvature dependent terms in the RHS of Eq. (2.2). For the derivation of the energy conditions in nonminimally coupled theory we follow closely Ref. [17].

In order to derive the NEC and the SEC, one usually considers the Raychaudhuri equation together with the request that gravity is attractive. In the case of a congruence of a time-like geodesic defined by a vector field u^μ this equation reads:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu, \quad (3.1)$$

where θ , $\sigma_{\mu\nu}$, $\omega_{\mu\nu}$ are, respectively, the expansion parameter, the shear and the rotation associated to the congruence.

For null geodesics k^ν , the Raychaudhuri equation is given by:

$$\frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu. \quad (3.2)$$

These equations are purely geometric and independent of the gravity theory we are considering. The connection with the gravity theory arises when we express Eqs. (3.1) and (3.2) in terms of $T_{\mu\nu}$. Assuming that gravity is attractive (convergence of geodesics) we have $\frac{d\theta}{d\tau} < 0$ and, since $\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$, for any hypersurfaces of orthogonal congruences (for vanishing rotation), we get the following conditions:

- *Strong-energy condition (SEC)*

$$R_{\mu\nu}u^\mu u^\nu \geq 0; \quad (3.3)$$

- *Null-energy condition (NEC)*

$$R_{\mu\nu}k^\mu k^\nu \geq 0. \quad (3.4)$$

In the GR case, one then uses Einstein's field equations into the NEC and SEC conditions (3.4) and (3.3). We get respectively:

$$T_{\mu\nu}k^\mu k^\nu \geq 0; \quad \left(T_{\mu\nu} - g_{\mu\nu} \frac{T}{2}\right) u^\mu u^\nu \geq 0. \quad (3.5)$$

On the other hand, in the case of the nonminimally coupled theory (2.1) we have to introduce an effective coupling constant κ and an additional energy momentum tensor $\hat{T}_{\mu\nu}$ describing the effects of the nonminimal coupling between matter and gravity.

We start by writing Eqs. (2.2) in the form:

$$G_{\mu\nu} = \kappa(\hat{T}_{\mu\nu} + T_{\mu\nu}), \quad (3.6)$$

where

$$\kappa = \frac{1 + \lambda f(R)}{1 + 2\mathcal{L}_m \lambda f_R(R)}, \quad (3.7)$$

which is an effective coupling constant, and

$$\hat{T}_{\mu\nu} = \frac{1}{[1 + \lambda f]} \left\{ -\lambda \mathcal{L}_m f_R R g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square)(2\lambda \mathcal{L}_m f_R) \right\}. \quad (3.8)$$

In order to keep gravity attractive, we have the additional condition $\kappa > 0$.

If we define \hat{T} as the trace of $\hat{T}_{\mu\nu}$, using Eqs. (3.6) into Eqs. (3.3) and (3.4), one obtains:

- *SEC condition*

$$R_{\mu\nu}u^\mu u^\nu = \left[\kappa(\hat{T}_{\mu\nu} + T_{\mu\nu}) + \frac{1}{2}g_{\mu\nu}(\hat{T} + T) \right] u^\mu u^\nu \geq 0; \quad (3.9)$$

- *NEC condition*

$$R_{\mu\nu}k^\mu k^\nu = \left[\kappa(\hat{T}_{\mu\nu} + T_{\mu\nu}) + \frac{1}{2}g_{\mu\nu}(\hat{T} + T) \right] k^\mu k^\nu = \kappa(\hat{T}_{\mu\nu} + T_{\mu\nu})k^\mu k^\nu \geq 0. \quad (3.10)$$

By choosing a suitable reference frame, one can always take a null vector of the form

$$k^\mu = (-1 \ 1 \ 0 \ 0). \quad (3.11)$$

Assuming $\kappa > 0$ the NEC conditions become

$$-(\hat{T}_0^0 + T_0^0) + (\hat{T}_1^1 + T_1^1) \geq 0. \quad (3.12)$$

We can write this condition in the form:

$$T_0^0 - T_1^1 \leq \frac{2\lambda}{(1 + \lambda f)} [(\nabla_1 \nabla^1 - \nabla_0 \nabla^0)(\mathcal{L}_m f_R)]. \quad (3.13)$$

The latter represents a generalization of the usual NEC of GR to nonminimally coupled theories.

A. NEC saturation

In GR, NEC saturation sets a strong constraint on the form of the matter sources. In fact, for GR saturation of the inequality (3.12) implies

$$T_0^0 = T_1^1. \quad (3.14)$$

The previous relation is satisfied, in particular, by matter sources of particular interest, on which this paper is focused: (a) The vacuum, (b) Electromagnetic charged sources and (c) Cosmological constant.

Upon using Einstein equations, Eq. (3.14) translates into a condition for the first two components of the Ricci tensor,

$$R_0^0 = R_1^1. \quad (3.15)$$

We can reformulate the previous statement in a different way. In GR, Eqs. (3.14) and (3.15) allow two independent ways to implement the saturation of the NEC: in terms of sources alone or in terms of the gravitational field alone (i.e. in the space of the solutions) and if we impose them simultaneously the field equation (2.5) becomes an identity.

The situation is somehow different in the case of the non minimally coupled theory. In this latter case, NEC saturation leads to the following condition:

$$\frac{[-\nabla_0^0 + \nabla_1^1](2\lambda f_R \mathcal{L}_m)}{1 + \lambda f} = T_0^0 - T_1^1. \quad (3.16)$$

Thus, in nonminimally coupled theories, NEC saturation is not a simple constraint on the form of the matter sources, but it leads to a relation between matter source and derivatives of the coupling function. Nonetheless, upon use of the fields equations, NEC saturation is equivalent to the same GR condition (3.15). In fact using Eqs. (3.13) and (2.5), one immediately finds Eq. (3.15).

These features have an important consequence, which we will use later for deriving static, spherically solutions of the nonminimally coupled theory. We cannot “trivialize” the field equation (2.5) by imposing the condition (3.15) and a condition on the sources. Given the presence of the term depending on $f(R)$ in Eq. (3.16), an additional condition involving the coupling function f is needed to transform Eq. (2.5) into an identity.

This also implies that, analogously to GR, in the nonminimally coupled theory, once we impose the condition (3.15), the NEC conditions (3.16) becomes an identity by virtue of the field equations.

IV. STATIC, SPHERICALLY SYMMETRIC SOLUTIONS

We are interested in static, spherically symmetric solutions of the field equations (2.2). We consider a static spacetime with spherical symmetry and use the following parametrization for the metric:

$$ds^2 = -e^{2\alpha} dt^2 + e^{-2\alpha} dr^2 + H(r)^2 d\Omega^2, \quad (4.1)$$

where $\alpha(r)$ and $H(r)$ are functions of the radial coordinate. Setting $H^2 = e^{2\rho}$, the non vanishing components of the Ricci tensor are:

$$\begin{cases} R_{00} = [\alpha'' + 2\alpha'^2 + 2\rho'\alpha']e^{4\alpha}, \\ R_{11} = -[\alpha'' + 2\alpha'^2 + 2\rho'' + 2\rho'^2 + 2\rho'\alpha'], \\ R_{22} = -[\alpha'\rho' + 2\rho'^2 + \alpha'\rho'' + \rho''']e^{2(\alpha+\rho)} + 1, \\ R_{33} = \sin^2\theta R_{22}. \end{cases} \quad (4.2)$$

where the prime denotes derivatives with respect to the radial coordinate r and $i = 0, 1, \dots, 3$ indicates respectively the coordinates t, r, θ, φ .

One can now easily write down the field equation (2.2) for the static, spherically symmetric case. We have three equations for only two independent metric functions ($\alpha(r)$ and $H(r)$). The system is in general overconstrained unless we treat $f(R)$ as a dynamical variable, i.e determined by the field equations. We will see later in detail that generalization to nonminimally coupled theory of the most interesting solution of GR imposes very weakly constraints on the form of $f(R)$, typically they will just constrain its value at some points.

As independent equations we can obviously choose linear combinations of the field Eqs. (2.2). In particular, we have

$$-R_0^0 + R_1^1 = -2e^{2\alpha-\rho} \frac{d^2 e^\rho}{dr^2}, \quad (4.3)$$

which used in Eq. (2.5) gives

$$\frac{2e^{2\alpha}}{[1+\lambda f]} \left[(1+2\lambda f_R \mathcal{L}_m) e^{-\rho} \frac{d^2 e^\rho}{dr^2} + \lambda \frac{d^2}{dr^2} (\mathcal{L}_m f_R) \right] = T_0^0 - T_1^1, \quad (4.4)$$

Let us now write the NEC (3.13) and the NEC saturation relation (3.15) in the static, spherically symmetric case. We obtain respectively:

$$T_0^0 - T_1^1 \leq \frac{2\lambda e^{2\alpha}}{(1+\lambda f)} \left[\frac{d^2}{dr^2} (\mathcal{L}_m f_R) \right], \quad (4.5)$$

and

$$\frac{d^2 e^\rho}{dr^2} = 0. \quad (4.6)$$

Up to irrelevant integration constants, we find from (4.6)

$$e^\rho = r. \quad (4.7)$$

Our main goal is the generalization to the nonminimally coupled theory (2.1), of the static, spherically symmetric solution of GR, namely the AdS spacetime, the SCHW and the RN black hole solutions, both in flat and AdS space. In order to derive these solutions, we use the following strategy. We will first write down the general condition that must be satisfied by the energy momentum tensor of the corresponding matter sources. The next step is the use of the NEC. All the above solutions satisfy the NEC in GR but, in principle, they allow for two kinds of generalizations in the nonminimal theory: (a) Solutions satisfying the NEC in the non-minimally coupled theory; (b) Solutions not satisfying the NEC in the non-minimally coupled theory. In general, these two classes of solutions have a complete different behaviour. The final step is integrating the field equations and, when possible, writing down the solution in an explicit form.

V. SOURCES CHARACTERIZED BY $T = \text{constant}$ AND $T_0^0 = T_1^1$

The simplest matter sources to be considered in the static, spherically symmetric case are those saturating the NEC in GR and characterized additionally by a constant trace of the energy momentum tensor, i.e

$$T_0^0 = T_1^1, \quad T = \text{constant} = a. \quad (5.1)$$

One can easily realize that conditions (5.1) are, in particular, satisfied by the Minkowski vacuum, by a cosmological constant and, away from the position of the source, by charged and uncharged pointlike sources of mass M .

Although the matter sources characterized by Eqs. (5.1) always saturate the NEC of GR (3.14), the corresponding solutions may or may not saturate the NEC for the NMC theory (4.5). Let us therefore discuss these two cases separately.

A. NEC saturating solutions

In this case the solutions and their matter sources must satisfy simultaneously Eqs. (5.1), (4.6) and saturate the NEC for non minimally coupled theories, Eq. (4.5). These solutions saturate both the NEC of GR and those of the NMC theories.

Inserting Eq. (5.1) into Eq. (4.5), we get the simple condition:

$$\frac{d^2}{dr^2}(\mathcal{L}_m f_R) = 0. \quad (5.2)$$

Eq. (5.2) can be easily solved to give,

$$(\mathcal{L}_m f_R) = \xi r + k, \quad (5.3)$$

where ξ and k are integration constants.

When $\xi \neq 0$ Eq. (5.3) yields a stringent constraint on the form of the coupling $f(R)$. On the other hand, for $\xi = 0$, it renders a very weak constraint on the form of $f(R)$. Let us discuss these two cases separately.

$$\boxed{\xi = 0}$$

In this case we have

$$\mathcal{L}_m f_R = k. \quad (5.4)$$

From Eq. (2.4) we get:

$$R = \frac{[1 + \lambda f(R)]a}{2\lambda k - 1}. \quad (5.5)$$

This equation implies that the scalar curvature of the spacetime must be necessarily constant $R = R_0$. The value of R_0 is obtained by solving Eq. (5.5).

An important consequence of having a spacetime of constant curvature is that the stress-energy tensor $T_{\mu\nu}$ is conserved. This follows immediately from Eq. (2.6). We have now two possible cases to examine:

(1) \mathcal{L}_m is not constant

In this case the only way to solve Eq. (5.4) is to set,

$$f_R(R_0) = k = 0. \quad (5.6)$$

and Eq. (5.5) gives

$$R_0 = -[1 + \lambda f(R_0)]a. \quad (5.7)$$

(2) $\mathcal{L}_m = \text{constant} = b$

In this case Eq. (5.4) gives

$$f_R(R_0) = \frac{k}{b}. \quad (5.8)$$

Similarly to the previous case, the scalar curvature R_0 is obtained by solving Eq. (5.5).

Since the scalar curvature is constant, in both cases (1) and (2), from Eq. (2.6) it follows immediately that for the solution belonging to this class, the energy-momentum tensor is covariantly conserved.

Constancy of the scalar curvature together with Eq. (5.4), implies, apart from constant covariance of $T_{\mu\nu}$, a drastic simplification of the field Eqs. (2.2):

$$R_{\mu\nu} = \frac{1}{(1+2\lambda k)} \left[(1+\lambda f(R_0))T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R_0 \right]. \quad (5.9)$$

By defining a rescaled stress-energy tensor and an “effective” cosmological constant,

$$\tilde{T}_{\mu\nu} = \left[\frac{1+\lambda f(R_0)}{1+2\lambda k} \right] T_{\mu\nu}, \quad \tilde{\Lambda} = \frac{1}{2} \frac{R_0}{1+2\lambda k}. \quad (5.10)$$

the previous equation takes the form of Einstein field equations of general relativity with stress energy tensor $\tilde{T}_{\mu\nu}$ and a cosmological constant $\tilde{\Lambda}$.

It is important to stress that the conditions for the existence of this class of solutions sets very weak constraints on the form of the coupling function $f(R)$. In fact equation (5.6) determines the value of the derivative of f in a point, whereas Eq. (5.8) does not impose any constraint on the form of $f(R)$ as the integration constant k remains undetermined.

$$\boxed{\xi \neq 0}$$

In opposition to the previous case, Eq. (5.3) gives stringent constraints on the form of $f(R)$. Together with the field equations (2.2) it gives a system of two independent differential equations for two variables $\alpha, f(R)$. Thus, in general the function $f(R)$ is completely determined, apart from integration constants, by the dynamics of the gravitational field.

Moreover, the function $f(R)$ is also constrained by the asymptotic behaviour of the scalar curvature R . One can easily show that solutions with flat or AdS asymptotics require a non-analytic behaviour of $f(R)$.

To show this fact, we examine the $r \rightarrow \infty$ behaviour of the fields and we assume that in this limit the curvature goes to a constant R_0 (with $R_0 = 0$ in the flat case). The asymptotic behaviour of \mathcal{L}_m can be found by assuming that matter fields in the action (2.1) have at most an infrared divergence due the infinite volume of the spacetime. Thus, at leading order in the $r \rightarrow \infty$, we have $\mathcal{L}_m = \mathcal{O}(1)$. With this, Eq. (5.3) gives for $r \rightarrow \infty$

$$f(R) = \mathcal{O}(r). \quad (5.11)$$

Assuming that $f(R)$ is analytic in R_0 we can expand it in power series of $R - R_0$. Substituting this into Eq. (5.11) one easily finds inconsistency with the assumed analyticity of $f(R)$ in R_0 .

In the following sections we shall consider explicit black hole solutions of our nonminimally coupled theory, for the case of the simplest, and most interesting, matter sources satisfying Eq. (5.1):

- (a) Maximally symmetric spaces;
- (b) Generalization of Schwarzschild black hole solution of GR ($T_{\mu\nu} = 0$ away from the source);
- (c) Charged black hole solutions in flat spacetime, the generalization of the Reissner-Nordstrom (RN) solution ($T_{\mu\nu} \neq 0, T = 0$);
- (d) Schwarzschild black hole solutions in AdS spacetime (SAdS);
- (e) RN solutions in AdS spacetime (RNADS).

Both solutions (d) and (e) are characterized by $T_{\mu\nu} \neq 0, T = \text{const} \neq 0$.

B. NEC non saturating solutions

In this case the solutions and their matter sources must satisfy Eqs. (5.1), but not Eqs. (4.6) and (4.5). Therefore they saturate the NEC of GR but not the NEC of nonminimally coupled theories. R is not constant and we get from Eq. (2.4)

$$R = \frac{-6\Box(\lambda\mathcal{L}_m f_R) + [1 + \lambda f(R)]a}{2\lambda\mathcal{L}_m f_R - 1}. \quad (5.12)$$

Moreover, the coupling function $f(R)$ is strongly constrained by the field equations. We get from Eq. (4.4)

$$\frac{1}{H(r)} \frac{d^2 H(r)}{dr^2} = - \left(\frac{\lambda}{1 + 2\lambda \mathcal{L}_m f_R} \right) \frac{d^2 (\mathcal{L}_m f_R)}{dr^2}, \quad (5.13)$$

Eqs. (5.13) and (5.12) together with the other independent equation in (2.2) give a system of three equations to be solved for the unknown functions $\alpha(r)$, $H(r)$ and $f(R)$. This system of differential equations is very difficult to solve in closed form. In the following sections we will advance some arguments ruling out the existence of solutions of this class.

In general the scalar curvature for this class of solutions is not constant. Thus, from Eq. (2.6) it follows that the energy-momentum tensor is not covariantly conserved.

Let us conclude this section with a remark on the uniqueness of black hole solutions in NMC theories of gravity. It is well known that in GR Birkhoff's theorem ensures that every spherically symmetric solution generated by a charged or uncharged pointlike mass is, up to spacetime diffeomorphisms equivalent to a static solution. Moreover, well-known no-hair theorems ensure that the SCHW and RN solution are unique [21].

By writing down the field equations of non minimally coupled gravity in the spherically symmetric, non-static case, one easily realizes that there is no evidence for a Birkhoff's theorem to hold, i.e spherically symmetric solutions are not necessarily static. For instance, one could have non static, spherically symmetric solutions that are not equivalent, modulo diffeomorphisms, to static solutions.

VI. COSMOLOGICAL CONSTANT

In this section we consider a source described by a cosmological constant, with a Lagrangian density given by Eq (2.8). The associated energy-momentum tensor

$$T_{\mu\nu} = -\Lambda g_{\mu\nu}, \quad T = -4\Lambda, \quad (6.1)$$

satisfies the condition (5.1), so that the considerations of Section V hold. Following the classification of Section V, we distinguish between solutions saturating or not the NEC. The first class of solutions gives origin to maximally symmetric spaces.

A. NEC saturating solutions and maximally symmetric spaces

As we have seen in Section V, analyticity of the coupling function requires $\xi = 0$ in Eq. (5.3) and we can therefore use the results of point (2) of section V A.

The scalar curvature is constant and determined by Eq. (5.9) and the field equation become the Einstein equations,

$$R_{\mu\nu} = \tilde{\Lambda} g_{\mu\nu}, \quad (6.2)$$

with an effective cosmological constant $\tilde{\Lambda}$ given by

$$\tilde{\Lambda} = \frac{R_0}{4} = \frac{1 + \lambda f(R_0)}{1 + 2\lambda \Lambda f_R(R_0)} \Lambda. \quad (6.3)$$

We can now solve Eq. (6.3) for Λ :

$$\Lambda = \frac{\tilde{\Lambda}}{1 + \lambda f(4\tilde{\Lambda}) - 2\tilde{\Lambda} \lambda f_R(4\tilde{\Lambda})}. \quad (6.4)$$

The solutions of Eq. (6.2) are maximally symmetric spaces with constant scalar curvature $R_0 = 4\tilde{\Lambda}$. Depending on the value of $\tilde{\Lambda}$ we can have the usual three cases: (1) $\tilde{\Lambda} = 0$, Minkowski space-time; (2) $\tilde{\Lambda} > 0$, dS space-time; (3) $\tilde{\Lambda} < 0$, AdS space-time.

The striking feature of the maximally symmetric spacetime in nonminimally coupled gravity is that the curvature of the spacetime is determined by an “effective” cosmological constant $\tilde{\Lambda}$ which is related in a nontrivial way to the bare cosmological constant Λ through the coupling function $f(R_0)$. Physically this means that curvature of the spacetime is determined not only by the density of energy of matter fields, but also by the coupling function $f(R)$. We shall get back to this important point in Section XI.

The AdS solution takes the form

$$e^{2\alpha} = 1 + \frac{r^2}{L^2}, \quad H = r, \quad (6.5)$$

where the AdS length L is expressed in terms of the effective cosmological constant by $L^2 = -3/\tilde{\Lambda}$.

B. NEC non saturating solutions

In this case we have to solve Eqs. (5.12) and (5.13). We consider only solutions that are asymptotically AdS, i.e $R - R_0 \sim r^{-\gamma}$, with $\gamma > 0$ and we use the $r \rightarrow \infty$ asymptotic expansion for $H(r)$: $H(r) = \sum_{n=\alpha} a_n r^n$. Under these assumptions, Eq. (5.13) gives,

$$\frac{1}{H(r)} \sum_n a_n n(n-1) r^{n-2} = \frac{1}{2} \left[\frac{2\lambda\Lambda}{1-2\lambda\Lambda f_R} \right] \frac{\partial^2}{\partial r^2} f_R. \quad (6.6)$$

Assuming that $f(R)$ is analytic in R_0 one can write $f_R(R) = f_R(R_0) + f_{RR}(R_0)r^{-\gamma} + \mathcal{O}(r^{-2\gamma})$, so that Eq. (6.6) gives

$$\begin{aligned} C\beta(\beta-1)\frac{1}{r^2} - \frac{C\beta(\beta-1)}{r^2} \left[2\lambda\Lambda \left(f_R(R_0) + f_{RR}(R_0) \left(\frac{1}{r^\gamma} \right) \right) + \mathcal{O} \left(\frac{1}{r^{2\gamma}} \right) \right] \sim \\ \sim \lambda\Lambda \frac{\partial^2}{\partial r^2} \left[f_R(R_0) + f_{RR}(R_0) \left(\frac{1}{r^\gamma} \right) + \mathcal{O} \left(\frac{1}{r^{2\gamma}} \right) \right], \end{aligned} \quad (6.7)$$

where Cr^β is the dominant term in $H(r)$. One can easily realize that the only way to solve the previous equation is by setting $\beta(\beta-1) = 0$, $(d^2 f_R/dr^2) = 0$, giving back the conditions of Eq. (5.2) for NEC saturation.

This means that the only solution of the NMC theory asymptotically consistent with the AdS spacetime is the AdS spacetime itself.

VII. GENERALIZED SCHWARZSCHILD SOLUTION

Let us now consider as source of the gravitational field a pointlike particle of mass M . In the following, when considering pointlike and extended sources to conform to the usual conventions we will use the Lagrangian $\mathcal{L}_m = -8\pi\rho$, where ρ is the mass density of the source. For the case of a pointlike particle of mass M the energy momentum tensor is therefore given by

$$T_{\mu\nu} = 8\pi\delta_{\mu 0}\delta_{\nu 0}M\delta(r). \quad (7.1)$$

In GR this gravitational source leads to the well known SCHW black hole solution.

In order to derive the solutions for the NMC theory (2.1) with source (7.1), we first observe that for $r \neq 0$, $\mathcal{L}_m = 0$ and the field equations (2.2) are the same as in GR. Hence the solution has the same form of the SCHW solution in GR,

$$e^{2\alpha} = 1 - \frac{2\tilde{M}}{r}, \quad H = r, \quad (7.2)$$

where \tilde{M} is an integration constant.

The difference between the SCHW solution of GR and that of the nonminimally coupled theory arises only when one looks at the relation between \tilde{M} and M . This can be found, as usual, by considering the weak-field, Newtonian limit of the field equations.

The weak field limit for a generic static, spherically symmetric, mass distribution with density $\rho(r)$ will be discussed in detail in Section IX. Here we consider only the case of a pointlike source, i.e. a source with a delta function singularity at $r = 0$. As for $r \neq 0$, $\mathcal{L}_m = 0$, we see that the only contributions to R and R_{00} come from the delta function singularity. Moreover, in the usual static, nonrelativistic, weak field limit of Eqs. (2.2) and (2.4) do not yield the Poisson equation unless the terms containing \mathcal{L}_m are identically zero. This can be simply achieved setting¹

$$f_R(R(r=0)) = 0. \quad (7.3)$$

Because $R \propto \delta(r)$, the previous Eq. (7.3) is equivalent to $f_R(R \rightarrow \infty) = 0$.

Using Eq. (7.3) we can perform the usual static, nonrelativistic, weak field limit in the field equations (2.2) and (2.4) and obtain Poisson equation for the Newtonian potential Φ ,

$$\nabla^2 \Phi = \frac{1}{2} [1 + \lambda f(R \rightarrow \infty)] T_{00}. \quad (7.4)$$

Whereas for GR, $f(R) = 0$, and Eqs. (7.2), (7.4) give $M = \tilde{M}$; for the NMC theory, one finds

$$\tilde{M} = [1 + \lambda f(R \rightarrow \infty)] M. \quad (7.5)$$

\tilde{M} has to be considered as the “effective” dressed gravitational mass of the black hole, whereas M is the “bare” one. In order to the two masses be the same:

$$f(R \rightarrow \infty) = 0. \quad (7.6)$$

Notice that the relation (7.5) yields a divergent mass if $f(R)$ diverges for $R \rightarrow \infty$.

Naively, one is lead to interpret \tilde{M} as the gravitational mass and M as the inertial mass of the pointlike particle and the relation (7.5) as an explicit breakdown of the equivalence principle. However, closer inspection reveals that this cannot be simply inferred by considering the motion of a test particle in a SCHW background. In fact, although it is true that equation of motion of a test particle of mass m in the SCHW metric contains a $f(R)$ -dependent connection, the R -dependent part cancels because $R = 0$, identically, whenever $r \neq 0$ and the motion is purely geodesic:

$$\frac{\delta S_m}{\delta x^\mu(\tau)} = m(1 + \lambda f(0)) \frac{\delta}{\delta x^\mu(\tau)} \int ds = 0. \quad (7.7)$$

We see that the coupling function does not effect the geodesic motion of the test particle. Moreover, Eq. (7.7) seems to tell us that the “inertial” mass of the test particle is equal to the dressed mass $\tilde{m} = m(1 + \lambda f(0))$. As long as we consider the motion of a test particle in a SCHW background there is no physical way to distinguish between M and \tilde{M} .

¹ This result can be derived along the same lines used in Section IX to derive the conditions for the existence of the Newtonian limit.

The distinction between bare and dressed mass and the observation of breakdown of the equivalence principle requires to go beyond the test particle approximation (for instance by considering the mutual interaction of two black holes), or to consider more general matter distributions (for instance a collapsing star). In this general situations $R \neq \text{const}$ and the equation of motion for matter (7.7) will get a $f(R)$ -dependent contribution to the connection.

A. Generalized SAdS solution

The asymptotically flat generalized SCHW solution (7.2) can be easily extended to the case in which a negative cosmological constant Λ is present. In GR one obtains in this way the asymptotically AdS black hole solutions (SAdS solutions).

Considering the case where the NEC is saturated, the SAdS solution is easily obtained from the AdS solution (6.5) by introducing a matter source of the form (7.1):

$$e^{2\alpha} = 1 + \frac{r^2}{L^2} - \frac{2\tilde{M}}{r}, \quad H = r, \quad (7.8)$$

where \tilde{M} is the effective gravitational mass (7.5) and $L^2 = -3/\tilde{\Lambda}$, $\tilde{\Lambda}$ being the effective cosmological constant (6.3).

VIII. CHARGED BLACK HOLE SOLUTIONS

In this section, we consider electromagnetically charged black hole solutions in the NMC theory. The matter-field action is obtained by using in Eq. (2.1) the Lagrangian density for the electromagnetic (EM) field:

$$\mathcal{L}_m = -F^2 = -F_{\mu\nu}F^{\mu\nu}. \quad (8.1)$$

The $F^{\mu\nu}$ is the EM tensor. The resulting field equations for the EM field are

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}(1 + \lambda f(R))) = 0. \quad (8.2)$$

We first consider the purely electrically charged case. Only the tr component of the electromagnetic tensor is non vanishing and the Maxwell equations (8.2) can be immediately solved to give

$$F_{01} = -F_{10} = -\frac{Q}{H^2(1 + \lambda f)}, \quad (8.3)$$

where Q is an integration constant, which in GR gives the electric charge of the solution. The energy-momentum for the EM field is given by,

$$T_{00} = 4\frac{Q^2}{H^4(1 + \lambda f)^2}e^{2\alpha}, \quad T_{11} = -e^{-4\alpha}T_{00}, \quad T_{22} = 4\frac{Q^2}{H^2(1 + \lambda f)^2}, \quad T_{33} = \sin^2\theta T_{22}. \quad (8.4)$$

Moreover using Eq. (8.3) one easily finds that

$$\mathcal{L}_m = -F^2 = 2(F_{01})^2 = \frac{2Q^2}{H^4(1 + \lambda f)^2}. \quad (8.5)$$

It can be checked that the energy momentum tensor (8.4) satisfies the conditions $T = 0$ and $T_0^0 = T_1^1 = 0$, which is a particular case of those discussed in Section V. The first condition expresses the conformal invariance of the EM action, while the second concerns the GR NEC saturation.

As before, we distinguish between the solutions that saturate NEC from the ones that do not in the NMC theory.

A. NEC saturating solutions

The general results of Section V A tell us that solutions characterized by $\xi \neq 0$ are ruled out due to the mismatch of the asymptotics and of the analyticity of $f(R)$. For charged solutions the conditions ruling out $\xi \neq 0$ hold even more strongly because from Eq. (8.5) we see that as $r \rightarrow \infty$, \mathcal{L}_m decays as $\mathcal{O}(r^{-4})$.

NEC saturating solutions are therefore obtained setting $\xi = 0$ and using Eqs. (8.5) and (5.6). For $b = T = 0$, we get from Eqs. (8.5) and (5.6)

$$R = 0, \quad f_R(0) = 0. \quad (8.6)$$

From this, the field equations (2.2) give

$$R_{00} = -\frac{4g_{00}Q^2}{r^4(1+\lambda f(0))}, \quad \frac{d}{dr}[re^{2\alpha}] = 1 - \frac{4Q^2}{r^2(1+\lambda f(0))}. \quad (8.7)$$

If we take $\lambda = 0$ we recover, as expected, the GR equations,

$$R_{00} = -\frac{4g_{00}Q^2}{r^4}, \quad \frac{d}{dr}[re^{2\alpha}] = 1 - \frac{4Q^2}{r^2}. \quad (8.8)$$

The system of Eqs. (8.7) becomes completely equivalent to (8.8) by defining an “effective”, dressed, charge

$$\tilde{Q} = \frac{Q}{\sqrt{1+\lambda f(0)}}. \quad (8.9)$$

It follows immediately that the charged black solutions of Eqs. (8.7) have the same form of the RN solution of GR, with the charge Q and mass M replaced respectively by the effective charge \tilde{Q} of Eq. (8.9) and by the effective mass \tilde{M} of Eq. (7.5):

$$ds^2 = -\left(1 - \frac{2\tilde{M}}{r} + \frac{4\tilde{Q}^2}{r^2}\right) dt^2 + \left(1 - \frac{2\tilde{M}}{r} + \frac{4\tilde{Q}^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (8.10)$$

The dressed and bare charges are the same when

$$f(R=0) = 0. \quad (8.11)$$

The electrically charged solution of the NMC gravity theory can be simply obtained from the RN black hole solutions of GR by a simple, coupling function dependent, charge and mass rescaling. This means that the physical effect of the nonminimal coupling on the charged static black hole solution is a redefinition of the physical parameters of the black hole, which contains the information about the local behaviour of the coupling function at $R = 0$ and $R = \infty$.

On the other hand, we see that solution (8.10) is allowed only if the coupling function $f(R)$ is constrained by Eq. (8.6). This is an important selection criteria for the coupling function.

It should be pointed out that, similarly to the dressed mass, as long as one considers only test particles the physically observable charge of the black hole is the dressed one \tilde{Q} of Eq. (8.9). This is immediately evident from the solution for the Maxwell field strength (8.3).

It is also of interest to compare the behaviour of charged black hole solutions of the NMC gravity theory with the charged black hole solutions of the Dilaton-Maxwell-Einstein gravity. In the latter theories, the charge rescaling effect is very similar to Eq. (8.9), with the rescaling factor determined by the asymptotic value of the dilaton [22, 23]. However, both the form and the causal structure of these solution are radically different from the RN solutions. Strangely enough the NMC theory allows for charged solutions which are a “minimal” deformation (just a charge and mass rescaling) of the RN solutions.

The occurrence of this minimal deformation can be also directly seen from Eq. (2.6). The stress energy tensor is covariantly conserved whenever $R = 0$.

B. Dual and dyonic solutions

Electromagnetic duality holds also in the NMC theory (2.1) with the matter Lagrangian density given by (8.1). In fact, the transformation $F \rightarrow \tilde{F}$ with $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ leaves the action (8.1) unchanged up to a sign and leads, therefore, to the same Maxwell equations. The duality transformation leaves the metric part of the solution unchanged and transforms the purely electric solution (8.3) into a purely magnetic monopole solution

$$F_{23} = \frac{P}{1 + \lambda f} \sin \theta, \quad (8.12)$$

where P is the magnetic charge. We can define, analogously to Eq. (8.9), the effective, dressed, magnetic charge

$$\tilde{P} = \frac{P}{\sqrt{1 + \lambda f(0)}}. \quad (8.13)$$

For sources carrying both electric and magnetic charges we have the dyonic charged black hole solution given by,

$$ds^2 = - \left(1 - \frac{2\tilde{M}}{r} + \frac{4(\tilde{Q}^2 + \tilde{P}^2)}{r^2} \right) dt^2 + \left(1 - \frac{2\tilde{M}}{r} + \frac{4(\tilde{Q}^2 + \tilde{P}^2)}{r^2} \right)^{-1} dr^2 + r^2 d\Omega^2. \quad (8.14)$$

C. Generalized RNAdS solutions

Similarly to the SAdS solution one can easily construct the generalization of the RNAdS solution of GR for the NMC theory. The matter Lagrangian density has now (apart from the mass M of the source localized in $r = 0$) two contributions, a cosmological constant and an electromagnetic term:

$$\mathcal{L}_m = -\Lambda - F^2. \quad (8.15)$$

The corresponding stress-energy tensor $T_{\mu\nu}$ will have two contributions and the purely electric, static, spherically symmetric solution, with $\xi = 0$, is given by

$$e^{2\alpha} = 1 + \frac{r^2}{L^2} - \frac{2\tilde{M}}{r} + \frac{4\tilde{Q}^2}{r^2}, \quad H = r, \quad (8.16)$$

where \tilde{M} is the effective gravitational mass Eq. (7.5), $L^2 = -3/\tilde{\Lambda}$, $\tilde{\Lambda}$ being the effective cosmological constant Eq. (6.3), and \tilde{Q} is the effective charge Eq. (8.9).

D. NEC non saturating solutions

The existence of charged solutions that do not saturate NEC can be investigated using a method similar to that used in Section VI B. We search for asymptotically flat solutions of Eqs. (5.12) and (5.13) with \mathcal{L}_m given by Eq. (8.1). We consider therefore $R \sim r^{-\gamma}$, with $\gamma > 0$ and we use the same $r \rightarrow \infty$ asymptotic expansion for $H(r)$, namely, $H(r) = \sum_{n=\beta} a_n r^n$.

Assuming that $f(R)$ is analytic in $R = 0$, Eq. (5.13) gives at leading order in $r \rightarrow \infty$:

$$\begin{aligned} \frac{\beta(\beta-1)}{r^2} + \frac{4\lambda Q^2 \beta(\beta-1)}{r^{4\beta+2}} \left[\frac{f_R(0)}{[1 + \lambda f(0)]^2} \right] + \mathcal{O} \left(\frac{1}{r^{\gamma+4\beta+2}} \right) \sim \\ \sim -2 \left[\frac{20\lambda Q^2}{r^{4\beta+2}} \frac{f_R(0)}{[1 + \lambda f(0)]^2} + \mathcal{O} \left(\frac{1}{r^{\gamma+4\beta+2}} \right) \right]. \end{aligned} \quad (8.17)$$

It follows immediately that, at leading order, we must have $\beta = 1$, $f_R(0) = 0$, which leads to the same solutions obtained in Section VIII A when NEC was saturated.

IX. WEAK FIELD LIMIT

In this section we discuss the weak field limit of the NMC theory (2.1) for the case of a static, spherically symmetric, extended source with rest mass density $\rho(r)$. Differently from GR, the weak-field limit of the gravity theory (2.1) is a rather involved issue. This is due to the fact that its feature depends crucially on the form of the coupling function $f(R)$. This is immediately evident if one considers the weak-field expansion near flat space. If the coupling function $f(R)$ diverges in $R = 0$ this weak field limit is meaningless. It is also possible that non minimally coupled gravity theories allow for alternative weak coupling expansions, for instance, near some background with nonvanishing curvature.

In this paper we assume the validity of the usual weak-field expansion around a flat space. We circumvent the above problems by assuming that the coupling function is analytic in $R = 0$, so that we can write,

$$f(R) = f(0) + f_R(0)R + \mathcal{O}(R^2). \quad (9.1)$$

We then use the usual weak-field, nonrelativistic, expansion of the metric around flat space-time

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}, \quad (9.2)$$

where $\eta_{\mu\nu}$ is the flat space-time metric tensor and $|h_{\mu\nu}| \ll 1$. We assume as usual that $h_{\mu\nu}$ is time independent, that the matter Lagrangian density is dominated by the mass density, $\mathcal{L}_m = -8\pi\rho(r)$ and that the density vanishes asymptotically, $\lim_{R \rightarrow 0} \rho = 0$.

In the weak field, nonrelativistic limit, the field equations are dominated by the the rest mass, so that the only relevant part is the tt -component. Expanding the coupling function $f(R)$ in power series, retaining only the linear terms in the curvature and using the following field expansions,

$$\nabla_\nu \nabla_\mu [\rho(f_{RR}(0)R)] \approx \partial_\mu \partial_\nu [\rho(f_{RR}(0)R)], \quad g_{\mu\nu} \square [\rho(f_{RR}(0)R)] \approx \eta_{\mu\nu} (f_{RR}(0) \frac{d^2(\rho R)}{dr^2}), \quad (9.3)$$

we get from the field equations (2.2).

$$[1 - 16\pi\lambda\rho f_R(0)]R_{00} - \frac{1}{2}g_{00}R = -2\lambda \frac{d^2}{dr^2} [8\pi\rho(f_R(0) + f_{RR}(0)R)] + 8\pi[1 + \lambda(f(0) + f_R(0)R)]\rho(r). \quad (9.4)$$

We now compute from Eq. (2.4) the scalar curvature in our approximation. We get

$$R \approx \frac{1}{1 + 16\pi\lambda\rho f_R(0)} \left\{ 8\pi(1 + \lambda f(0))\rho - 48\pi\lambda f_{RR}(0) \frac{d^2(\rho R)}{dr^2} - 48\pi\lambda f_R(0) \frac{d^2\rho}{dr^2} \right\}. \quad (9.5)$$

Retaining only the linear terms in ρ we have

$$R \approx 8\pi(1 + \lambda f(0))\rho - 48\pi\lambda f_R(0) \frac{d^2\rho}{dr^2}. \quad (9.6)$$

Using the usual static, non relativistic, weak field expression $R_{00} \approx -\frac{1}{2}\nabla^2 h_{00}$, to leading order in the curvature and in ρ , Eq. (9.4) becomes

$$\nabla^2 h_{00} \approx -16\pi\lambda f_R(0) \frac{d^2\rho}{dr^2} - 8\pi[1 + \lambda f(0)]\rho(r). \quad (9.7)$$

We see that for generic coupling function $f(R)$ and generic matter distribution the previous equation is not the usual Poisson equation, i.e the weak field limit of the NMC theory is not the usual

Newtonian gravity. If we exclude the particular matter distribution $\rho(r) = ar + b$ (not compatible with the boundary condition $\lim_{R \rightarrow 0} \rho = 0$), the only way to recover the Newtonian limit is to constrain the value of $f_R(0)$ as in Eq. (8.6), i.e $f_R(0) = 0$.

Once the constrain Eq. (8.6) is used, Eq. (9.7) takes the usual Poisson form

$$\nabla^2 h_{00} = -8\pi \tilde{\rho}(r), \quad (9.8)$$

if one defines a coupling function dressed mass density given by

$$\tilde{\rho} = [1 + \lambda f(0)]\rho(r). \quad (9.9)$$

The condition for the equality of the dressed and bare rest mass densities is $f(0) = 0$, i.e the same conditions (8.11) for the equality of dressed and bare black hole charges. This is rather unexpected. Completely uncorrelated results, namely the existence of charged black hole solutions and the equality $Q = \tilde{Q}$ on one side, and the existence of the usual Newtonian limit and the equality $\tilde{\rho} = \rho$ on the other side, can be achieved using the same conditions (8.6), (8.11).

Notice that both the conditions for the existence of the usual weak field Newtonian limit Eq. (8.6) and the definition of dressed mass density (9.9) are, respectively, analogous to the condition for the existence of the Newtonian limit for a pointlike source (7.3) and to the definition of dressed black hole mass given by Eq. (7.5). There is, however, a crucial and important difference. For the pointlike particle, the existence conditions of the Newtonian limit constrain the coupling function in the UV (at $r = 0$ or equivalently at $R \rightarrow \infty$) and the dressing of the black hole mass is realized in terms of the behaviour of the coupling function $f(R)$ in the same UV region. On the other hand for a spherically symmetric extended source considered in this section, the condition of existence and the dressing are characterized by the behaviour of the coupling function in the IR (i.e at $r \rightarrow \infty$ or equivalently, $R = 0$). This is a rather natural feature if one considers the localization of the pointlike source.

X. THE COUPLING FUNCTION

The NMC gravity theory (2.1) has to be considered as an effective description of some, yet to be discovered, fundamental theory of gravity. The coupling function contains presumably information about the behaviour of the fundamental theory in its various dynamical regimes. In the previous sections we have found that the existence of black holes solutions and the form of the physical parameters associated to them gives local constraints on the coupling function $f(R)$. This information can be used both for selecting a form of the coupling function $f(R)$ or, eventually, falsify the theory.

It is also important to stress that the information about the coupling function $f(R)$ inferred from the existence of black holes has a strong heuristic power as it covers the full range of energy scales of the gravitational interaction. In fact, we have both constraints on $f(R)$ in the, $R \rightarrow \infty$, strong curvature region (corresponding to the ultraviolet, quantum gravity regime) and constraints on $f(R)$ for, $R = 0$, weak curvature region (corresponding to the infrared regime) of the gravitational interaction.

The information about $f(R)$ that can be gathered from our previous results about static, spherically symmetric solutions of the NMC theory can be organized in two classes: (a) Constraints on $f(R)$ emerging from the requirement of existence of solutions; (b) constraints arising from the form of black hole dressed parameters.

A. Constraints on $f(R)$ arising from the existence of solutions

Existence of the Minkowski space vacuum solution does not set any constraint on $f(R)$. On the other hand, from Eq. (6.3) the existence of the dS and AdS solution constrains the IR behaviour

of the coupling function $f(R)$, which must be finite, as well as its first derivative, when evaluated on R_0 (spacetime curvature corresponding to the cosmological constant).

The existence of the SCHW solutions constrains the UV behaviour of $f(R)$. From Eq. (7.3) and (7.5) we see that $f_R(R \rightarrow \infty) = 0$ and $f(R \rightarrow \infty)$ must be finite. On the other hand, the existence of the RN solutions (8.10) requires that $f_R(R = 0) = 0$, and from (8.9), that $f(R = 0)$ is finite, constraining the deep IR behaviour of the coupling function.

Considering both pointlike and extended, spherically symmetric sources, the existence of the Newtonian weak field limit constrains the coupling function $f(R)$ both at the UV and at IR. For pointlike sources, we have $f_R(R \rightarrow \infty) = 0$. For extended sources we have $f_R(R = 0) = 0$, the same conditions for the existence of the RN solution.

B. Constraints arising from the dressed parameters

The existence of the relations (7.5), (8.9), (6.3) and (9.9) between bare and dressed black hole observables has a strong predictive power. In fact, by measuring gravitationally dressed parameters $\tilde{\Lambda}, \tilde{M}, \tilde{Q}, \tilde{\rho}$ and the bare ones one could either get informations on the coupling function $f(R)$ or falsify the theory.

Assuming that we can measure independently both the dressed and bare parameters, these relations can be used to restrict the form of the coupling function $f(R)$ at different dynamical regimes.

Let us consider as an example the situations in which all the standard spherically symmetric solutions of GR (SCHW, RN, dS, AdS, SADS, RNAdS) fully coincide with those of the NMC theory, i.e all the dressed parameters turn out to be equal to the bare ones. A solution of the equations $M = \tilde{M}$, $Q = \tilde{Q}$, $\Lambda = \tilde{\Lambda}$, $\tilde{\rho} = \rho$ is given by

$$\begin{aligned} f_R(R \rightarrow \infty) &= 0, & f(R \rightarrow \infty) &= 0, & f_R(R = 0) &= 0, \\ f(R = 0) &= 0, & f_R(R = R_0) &= 0, & f(R = R_0) &= 0. \end{aligned} \quad (10.1)$$

Eq. (10.1) give conditions on the local behaviour of the coupling function $f(R)$ at different length scales. The first two conditions, which arise from existence of the SCHW solution and from $M = \tilde{M}$, essentially tells us that quantum gravity effects can be described by the NMC theory as a perturbation of GR. Conversely, the third and fourth conditions in (10.1), arising from the existence of the RN solutions and from $\tilde{Q} = Q$, imply the same, but in the IR: any infrared modification of GR can be formulated as a perturbative expansion of GR.

Finally, the fifth and sixth conditions in (10.1), arising from $\tilde{\Lambda} = \Lambda$ allows us to expand perturbatively the NMC theory near GR at some, intermediate, cosmological length scale.

An observation of a mismatch between bare and dressed parameter would be a strong indication of the validity of the nonminimally coupled theory.

Notice that the solution of the equation $\Lambda = \tilde{\Lambda}$ given by the last two equations in (10.1) is not unique. In fact, using (6.4) one finds that this equation is generally solved by $2f(R_0) = R_0 f_R(R_0)$.

Apart from the existence and behaviour of black hole solutions, the form of the coupling function $f(R)$ can be constrained by several phenomenological considerations. Two of them are of particular relevance (a) Experimental constraints on the post-Newtonian parameters; (b) In order to keep gravity attractive, assuming $\lambda f(R) \ll 1$, then $f_R(R) < \frac{1}{2\lambda}$.

The experimental constraints on the post-Newtonian parameter have been discussed in Ref. [18] starting with a more general action given by: $S = \int [f_1(R) + f(R)\mathcal{L}] \sqrt{-g} d^4x$ and assuming definite, albeit rather general form for the coupling function $f(R)$.

As an example of implementation of the previous conditions, we consider here a generalization of the coupling function proposed in Ref. [18], which is suitable to satisfy the conditions for the existence of the various black hole solutions of the nonminimally coupled theory. We take

$$f(R) = \sum_{n>0}^N \frac{r_n^{2n}}{R^{2n} + C_n} + K, \quad (10.2)$$

where r_n , C_n and K are parameters, and $K = f(R \rightarrow \infty)$.

One can easily see that this form of $f(R)$ identically satisfies the condition for the existence of the SCHW solution (7.3), the RN solution (8.6) (hence also for the existence of the Newtonian limit for an extended, spherically symmetric source). Moreover, using the coupling function (10.2) the conditions (10.1) for the equality between dressed and bare parameters can be easily implemented in terms of the relation between the coefficients appearing in Eq. (10.2).

The equality between \tilde{M} and M can be easily obtained setting $K = 0$. On the other hand, the equality between \tilde{Q} and Q (hence the equality between $\rho = \tilde{\rho}$) requires

$$f(0) = \sum_n \frac{r_n^{2n}}{C_n} + K = 0. \quad (10.3)$$

Finally, introducing Eq. (10.2) into Eq. (6.3) one finds that the equality $\tilde{\Lambda}$ and Λ can be satisfied by requiring

$$C_n = -(n+1)R_0^{2n}. \quad (10.4)$$

C. Other possible scenarios

The scenario described in Section XB is a rather conservative one. $f(R)$ is assumed to be analytic and all the conditions for the existence of static spherically symmetric solutions and of the Newtonian limit are satisfied and, additionally, the conditions (10.1) for the equality of bare and dressed parameters also hold. Thus, at all the relevant physical scales (UV, cosmological, far IR) the gravitational interaction can be described as perturbative expansion of the nonminimally coupled theory (2.1) in terms of powers of the curvature, with GR being the leading approximation.

Obviously, this is not the only logical possible scenario. There are several other possibilities. The first one is that we can retain analyticity of $f(R)$, but at least some of the equalities (10.1) are not satisfied. If this is the case, we still have a perturbative expansion around GR, but we would observe a modification of the spherically symmetric solutions of GR and/or of the weak field limit.

The other possibilities are related to a failure of analyticity of $f(R)$ at some dynamical regime. This failure can occur (a) at the UV (at $r = 0$); (b) at some intermediate scale l ; (c) in the far IR ($r \rightarrow \infty$). One of the most important results of this paper is the claim that this failure does have a direct impact on the existence of static, spherically symmetric solutions. If $f(R)$ diverges at $R \rightarrow \infty$ this implies that the SCHW solution in the nonminimally coupled theory does not exist. Physically, this means that if quantum gravity effects cannot be described, perturbatively, by a theory (2.1) the SCHW black hole solution cannot exist. Conversely, a divergence of $f(R)$ in the far IR (i.e. at $r \rightarrow \infty$), implying that we cannot describe, perturbatively, infrared modification of gravity using a theory (2.1), would imply that both the RN solution and of the usual Newtonian limit are not admitted.

The most interesting application of the gravity theory (2.1) is to describe infrared modifications of gravity in order to mimic dark matter [12, 13]. This is the case described by point (b) above. We generically require analyticity of $f(R)$ at some intermediate scale. An interesting choice for $f(R)$ is an inverse power law [14]:

$$f(R) = \left(\frac{R_s}{R} \right)^n, \quad n > 0. \quad (10.5)$$

The coupling function (10.5) is analytic at every intermediate scale, for which the curvature $R = R_0$, but it diverges for $R = 0$. Therefore, one can naively expect that both the RN and the Newtonian

limit, for an extended spherically symmetric source, not to exist. However, this may not be necessarily the case. First, the form (10.5) could be an approximation that holds near $R = R_0$, whereas the exact form of $f(R)$ could be perfectly regular at $R = 0$. Second, all the results of this paper about the weak field limit have been derived assuming the usual weak field expansion around flat space ($R = 0$). An alternative weak field expansion, near some curved background can yield weak field features distinct from the ones discussed in this paper (see eg. Ref. [18]).

XI. APPLICATION TO THE COSMOLOGICAL CONSTANT PROBLEM

In this section we consider the dressing of the cosmological constant through the coupling function $f(R)$ given by Eq. (6.3) to address the cosmological constant problem [24–27].

In its standard formulation, the cosmological constant problem is the difficulty to explain why the present value of the cosmological constant, inferred from the universe acceleration data, is 120 orders of magnitude smaller than its natural value, inferred from microscopic physics ($\Lambda \sim M_p^2$, M_p being the Planck mass).

The dressing of the cosmological constant (6.3) in the NMC does seem to provide a natural way to adjust the cosmological constant at the level of the effective theory of gravity. The nonminimally coupled theory has to be considered as an effective theory and the coupling function $f(R)$ encodes the information about the gravitational dressing of all the physical parameters. In particular, in Eq. (6.3), the bare cosmological constant Λ describes the vacuum energy of matter fields, whereas the dressed cosmological constant $\tilde{\Lambda}$ is the effective gravitating vacuum energy of the matter fields.

The actual value of Λ depends on microphysics, it can range from the (TeV scale)² to M_p^2 . Here we assume $\Lambda \sim M_p^2$.

The key question, which we address in this section is: is there a choice of the coupling function $f(R)$, which on the one side allows to achieve in a natural way $\Lambda = 10^{120}\tilde{\Lambda}$ and on the other side is compatible with the constraints discussed in Section X.

By natural, we mean that the hierarchy of scales is achieved without any fine tuning and without introducing any other mass scale apart from $\tilde{\Lambda}$ (or Λ). It should also be noted that the constraints discussed in Section X are only a subset of the full set of conditions that the coupling function must satisfy such as, for instance, consistency with solar system data, the post-Newtonian approximation, gravitational lensing and so on. For sure, this is a limitation of the discussion of this section about the form of $f(R)$.

We first point out that a coupling function of the form (10.5) does not do allow for generating a ratio (see also discussions in Refs. [19, 20]),

$$\frac{\Lambda}{\tilde{\Lambda}} \sim 10^{120}. \quad (11.1)$$

In fact, introducing Eq. (10.5) into Eq. (6.4), one obtains

$$\frac{\Lambda}{\tilde{\Lambda}} = \frac{1}{1 + \lambda(1 + \frac{n}{2}) \left(\frac{R_s}{4\tilde{\Lambda}}\right)^n}, \quad (11.2)$$

which for a positive λ is evidently incompatible with (11.1). The situation is not much better if we take instead of Eq. (10.5) $f(R)$ as a positive power $f = (R/R_s)^n$. In this case, we get the same Eq. (11.2), but with the opposite sign of n . Thus, for $n > 2$ (11.2) becomes compatible with Eq. (11.1) but, nevertheless accurate fine tuning is needed to achieve (11.1). For instance, for $n = 4$ we obtain (λ is here irrelevant and has been set to 1), $1 - (4\tilde{\Lambda}/R_s)^4 = 10^{-120}$. Accurate fine tuning of the parameter R_s is needed in order to satisfy this equation.

Similar arguments rule out coupling functions with polynomial or rational form like (10.2), leaving as possible candidates coupling functions with exponential form.

Let us assume that in the region of small spacetime curvatures $R \sim \tilde{\Lambda}$ the coupling function f is well described by the function:

$$\lambda f(R) = e^{-\sigma R} - 1, \quad (11.3)$$

where σ is a positive parameter to be determined in order to achieve the hierarchy (11.1). This coupling function satisfy Eqs. (7.3), but not Eq. (7.6) and (8.6). As such, it allows for the existence of the SCHW solution but not for the existence of the RN solution and of the Newtonian limit for an extended, spherically symmetric source. Moreover, because Eq. (7.6) is not satisfied, the dressed mass of the SCHW black hole does not coincide with its bare mass.

One can easily show that the coupling function (11.3) allows for generating the hierarchy (11.1) without any unnatural fine tuning of the parameter σ . In fact, introducing Eq. (11.3) into Eq. (6.4) one gets,

$$\frac{\Lambda}{\tilde{\Lambda}} = \frac{e^{4\sigma\tilde{\Lambda}}}{1 + 2\sigma\tilde{\Lambda}}. \quad (11.4)$$

Choosing in Eq. (11.3) the dimensional constant σ as $\sigma \approx 70/\tilde{\Lambda}$, i.e just two orders of magnitude bigger then $1/\tilde{\Lambda}$, one manages to satisfy Eq. (11.1).

The physical mechanism behind the generation of the hierarchy (11.1) can be also understood by calculating the effective gravitational coupling κ , Eq. (3.7). In the region of small spacetime curvature, where the form of $f(R)$ (11.3) holds, we naturally expect the density of usual matter to be one or two orders of magnitude smaller then that of dark energy. We can therefore realistically assume $\mathcal{L}_m \sim 10^{-2}\tilde{\Lambda}$ so that we have,

$$\kappa \sim (1 + \lambda f(R_0)) \sim e^{-4\sigma\tilde{\Lambda}} \sim 10^{-120}. \quad (11.5)$$

This means that the effective cosmological constant is so tiny compared to the expected vacuum energy of matter fields, because the effective coupling constant (3.7) becomes extremely tiny in the regions of small spacetime curvature.

Although appealing from several points of view, the choice of the coupling function (11.3) is not completely satisfactory. The main problem is that it does not allow for the existence of the RN solution and of the Newtonian limit for extended, spherically symmetric sources. This is particularly disturbing because the coupling function (11.3) is assumed to be a good description in the regions of small R , the same region relevant for the above existence conditions.

We can do better by assuming that in the region of small spacetime curvatures, f is given by

$$\lambda f(R) = e^{-\sigma^2 R^2} - 1. \quad (11.6)$$

This choice for the coupling function allows to satisfy not only Eq. (7.3) but also Eq. (8.6). This means that, with this choice of $f(R)$, the existence of the the SCHW solution, of the RN solution and of the Newtonian limit for an extended, spherically symmetric source is guaranteed. Since Eqs. (8.11) and (7.6) do not hold with the choice (11.6), dressed masses, charges and mass densities are different from the bares ones.

With $f(R)$ given by Eq. (11.6), Eq. (6.4) gives,

$$\frac{\Lambda}{\tilde{\Lambda}} = \frac{e^{16\sigma^2\tilde{\Lambda}^2}}{1 + 16\sigma^2\tilde{\Lambda}^2}. \quad (11.7)$$

Choosing in (11.6) the dimensional constant σ of the same order of magnitude of $1/\tilde{\Lambda}$: $\sigma \sim 4.2/\tilde{\Lambda}$, one reproduces the hierarchy of mass scales (11.1).

XII. CONCLUSIONS

In this work, we have sought for black hole solutions in an alternative theory of gravity with an explicit non-minimal coupling between matter and curvature. General black hole solutions satisfying the known energy conditions were considered including the ones with anti-de Sitter

background. Of particular relevance is the NEC, whose saturation or not, was shown to be a useful criterion to study general classes of solutions.

We have shown that in what concerns the usual black hole type solutions (Schwarzschild and Reissner-Nordstrom), the NMC affects very weakly the features of the black hole. The analytical form of the solutions remains as those of GR but with "bare" GR mass, charge and cosmological constant, are replaced by the corresponding "dressed" quantities which acquire a contribution from the non minimal coupling at a relevant curvature scale (c.f. sections VI, VII and VIII).

We have fall short of presenting a rigorous prove of the uniqueness of the static solutions in NMC theories analogous to Birkhoff's theorem in GR. However, we have shown that under reasonable assumptions on the analyticity of $f(R)$ and on the behaviour of the weak field limit the black hole solutions we have found in this paper are essentially unique.

We have also shown that the existence of black hole solutions as well as the conditions for a suitable weak field limit constitute a relevant constraint on the coupling function, $f(R)$, which sets the strength of the nonminimal coupling between matter and curvature. These constraints are particularly interesting as they provide information both on the local behavior of $f(R)$ near the singularity - where quantum gravity effects are expected to dominate - and at the asymptotic region - where gravity is described by the Newtonian weak field limit. The knowledge about the coupling function acquired in this paper, together with other constraints coming from previous investigations on large scale behaviour [12–14, 16], represent a useful guide for future investigation on theories of gravity with NMC between matter and curvature.

Finally, we have shown that the dressing of the bare cosmological constant trough the coupling function $f(R)$ can be used to generate in a natural way a hierarchy of 120 orders of magnitude between the bare and the dressed cosmological constant. Since we are working with an effective theory of gravity, this cannot be seen as a full solution of the cosmological constant problem. Nevertheless, it clearly shows that NMC theories of gravity can mimic in a very efficient way exotic forms of energy, most particularly when the solution of the problem requires bridging the short and large scale behaviour.

An important issue that we have not addressed in this paper, which deserves future investigation, is the possibility of existence of a weak field limit different from the usual one. Existence of the usual Newtonian weak field limit has played a crucial role in our investigation of black hole solutions in NMC theories. Furthermore, in combination with analyticity of the coupling function $f(R)$, it is essential to argue about the uniqueness of the black hole solutions. Moreover, its existence gives local constraints on $f(R = 0)$.

The existence of the usual Newtonian limit is a very natural requirement for black holes. This is not necessarily true if one considers different regimes of the gravitational interaction, where the existence of non-Newtonian weak field limit could represent an alternative to dark matter [28, 29].

In principle, NMC matter-curvature theories of gravity offer a suitable framework for an alternative scenario in which different regimes of the gravitational interaction are described by a different weak field limit and by a different local behaviour of the coupling function $f(R)$ (c.f. section X C).

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- [1] C. M. Will, *Living Rev. Relativity*, **17**, 4 (2014).
 - [2] O. Bertolami and J. Páramos, *Springer Spacetime Handbook (2014)*, arXiv:1212.2177 [gr-qc].
 - [3] S. Capozziello, V. F. Cardone and A. Troisi, *JCAP* **08**, 001 (2006).
 - [4] T. P. Sotiriou and V. Faraoni, *Rev. Mod. Phys.* **82**, 451 (2010).
 - [5] A. De Felice and S. Tsujikawa, *Living Rev. Relativity* **13**, 3 (2010).
 - [6] O. Bertolami, C. G. Böhrer, T. Harko and F. S. N. Lobo, *Phys. Rev. D* **75**, 104016 (2007).
 - [7] L. Amendola and D. Tocchini-Valentini, *Phys. Rev. D* **64**, 043509 (2001).
 - [8] G. Allemandi, A. Borowiec, M. Francaviglia and S. D. Odintsov, *Phys. Rev. D* **72**, 063505 (2005).
 - [9] O. Bertolami and J. Páramos, *Phys. Rev. D* **77**, 084018 (2008).
 - [10] J. Páramos and C. Bastos, *Phys. Rev. D* **86**, 103007 (2012).
 - [11] O. Bertolami, P. Frazão and J. Páramos, *Phys. Rev. D* **83**, 044010 (2011).
 - [12] O. Bertolami and J. Páramos, *JCAP* **03**, 009 (2013).
 - [13] O. Bertolami, P. Frazão and J. Páramos, *Phys. Rev. D* **86**, 044034 (2012).
 - [14] O. Bertolami, P. Frazão and J. Páramos, *Phys. Rev. D* **81**, 104046 (2010).
 - [15] O. Bertolami, P. Frazão and J. Páramos, *JCAP* **05**, 029 (2013).
 - [16] O. Bertolami and J. Páramos, *Int. J. Geom. Meth. Mod. Phys.* **11**, 1460003 (2014).
 - [17] O. Bertolami and M. C. Sequeira, *Phys. Rev. D* **79**, 104010 (2009).
 - [18] O. Bertolami, R. March and J. Páramos, *Phys. Rev. D* **88**, 064019 (2013).
 - [19] O. Bertolami and J. Páramos, *Phys. Rev. D* **84**, 064022 (2011).
 - [20] O. Bertolami and J. Páramos, *Phys. Rev. D* **89**, 044012 (2014).
 - [21] W. Israel, *Phys. Rev.* **164**, 1776 (1967); *Comm. Math. Phys.* **8**, 245 (1968).
 - [22] D. Garfinkle, G.T. Horowitz and A. Strominger, *Phys. Rev. D* **43**, 3140 (1991).
 - [23] M. Cadoni and S. Mignemi, *Phys. Rev. D* **48**, 5536 (1993).
 - [24] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).
 - [25] S. M. Carroll, *Living Rev. Rel.* **4**, 1 (2001).
 - [26] T. Padmanabhan, *Phys. Rept.* **380**, 235 (2003).
 - [27] O. Bertolami, *Int. J. Mod. Phys. D* **18**, 2303 (2009).
 - [28] M. Cadoni, *Gen. Rel. Grav.* **36**, 2681 (2004).
 - [29] M. Cadoni and M. Casula, *Gen. Rel. Grav.* **42**, 103 (2010).